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Baire's category and the bang-bang property for evolution differential inclusions of contractive type

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ABSTRACT

The Baire category method is employed in order to establish that the bang-bang property holds for a class of evolution differential inclusions of contractive type in reflexive and separable real Banach spaces.

1. Introduction

Bang-bang and relaxation type properties for evolution differential inclusions in Banach spaces have been studied from different points of view by several authors. Basic information and references can be found in the monographs of Aubin and Cellina [1] and Hu and Papageorgiou [21]. In most investigations a crucial role is played by the assumption that the right-hand side satisfies a globally Lipschitz type condition in the state variable (mere continuity is not sufficient as is shown by Pliś counterexample [31]). Under the weaker assumption that the right-hand side satisfies a locally Lipschitz condition in the state variable, no result seems available in infinite dimension. The aim of the present paper is to investigate this case in a rather general setting and, moreover, in the absence of any compactness assumption.

In our approach, we shall use the Baire category method. This was introduced in 1982 by De Blasi and Pianigiani [9–11] (starting from a generic type result proved by Cellina [5]) in order to study the existence of solutions of some classes of non-convex valued differential inclusions in Banach spaces, without hypotheses of compactness. Subsequently, the Baire method has been employed in different contexts by several authors including Bressan and Colombo [2], Papageorgiou [27], Suslov [32] for ordinary differential inclusions, and Bressan and Flores [3], Dacorogna and Marcellini [8], De Blasi and Pianigiani [12,13] for partial differential inclusions. An account of results obtained by means of the Baire method and a view on some recent problems concerning differential inclusions can be found in Pianigiani [30] and Cellina [6]. For a different method of approach to some of the above mentioned problems, making use of Gromov convex integration theory [20], see the contributions of Müller and Sverak [24] and Müller and Sychev [25]. A comparison of the two methods can be found in Sychev [33].

To describe the problem we want to study, let us introduce some notations. Let $\mathbb{E}$ be a real reflexive and separable Banach space and let A be the infinitesimal generator of a C_0-semigroup of contractions $T(t)$, $t \geqslant 0$, on $\mathbb{E}$. Let $I = [t_0, t_1]$ and let F be a multifunction defined on $I \times \mathbb{E}$ with nonempty closed convex bounded values $F(t,x) \subset \mathbb{E}$. Suppose further

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that F is continuous bounded and satisfies a locally Lipschitz condition in the x-variable. For $a \in \mathbb{E}$, consider the following convex and non-convex Cauchy problems:

$$\dot{x}(t) \in Ax(t) + F\big(t, x(t)\big), \quad x(t_0) = a, \tag{$C_{A,F,a}$}$$

$$\dot{x}(t) \in Ax(t) + \operatorname{ext} F\big(t, x(t)\big), \quad x(t_0) = a, \tag{$C_{A,\operatorname{ext} F,a}$}$$

where $\operatorname{ext} F(t, x(t))$ stands for the set of the extreme points of $F(t, x(t))$. Denote by $\mathcal{M}_{A,F,a}$ and $\mathcal{M}_{A,\operatorname{ext} F,a}$ the set of all mild solutions $x : I \to \mathbb{E}$ of $(C_{A,F,a})$ and $(C_{A,\operatorname{ext} F,a})$, respectively, and let u_x denote the pseudoderivative of x. Under the above assumptions $\mathcal{M}_{A,F,a}$, equipped with the metric of uniform convergence, is a nonempty complete metric space (with F merely continuous $\mathcal{M}_{A,F,a}$ could be empty by a result of Godunov [19]). Then the following bang-bang result holds true (Theorem 5.1):

$$\overline{\mathcal{M}}_{A,\operatorname{ext} F,a} = \mathcal{M}_{A,F,a}, \tag{1.1}$$

where the closure is in the metric of $\mathcal{M}_{A,F,a}$. In order to prove that, we define the sets

$$\mathcal{M}_n = \left\{ x \in \mathcal{M}_{A,F,a} \;\middle|\; \int_I d_F\big(t, x(t), u_x(t)\big)\, dt < 1/n \right\}, \quad n \in \mathbb{N},$$

where d_F is the Choquet function (which measure somehow the distance of $u_x(t)$ from $\operatorname{ext} F(t, x(t))$), and we show that each set $\mathcal{M}_n$ is open and dense in $\mathcal{M}_{A,F,a}$. Then, by the Baire theorem, the set

$$\mathcal{M}_0 = \bigcap_{n=1}^{\infty} \mathcal{M}_n$$

is residual in $\mathcal{M}_{A,F,a}$. Since, by the properties of the Choquet function, we have $\mathcal{M}_0 \subset \mathcal{M}_{A,\operatorname{ext} F,a}$, it follows that $\mathcal{M}_{A,\operatorname{ext} F,a}$ is dense in $\mathcal{M}_{A,F,a}$ and so (1.1) is valid.

The major and rather difficult part in the above plan is to show that each set $\mathcal{M}_n$ is dense in $\mathcal{M}_{A,F,a}$. In the classical approach to relaxation and bang-bang type properties for differential inclusions, a crucial role is played by the Filippov–Ważewski theorem which furnishes, for a given Cauchy problem with $F(t, x)$ globally Lipschitzian in the x-variable, an a priori global estimate of the distance of an approximate solution and its derivative from an exact solution and its derivative. For evolution differential inclusions $(C_{A,F,a})$, with $F(t, x)$ locally Lipschitzian in the x-variable and corresponding solution set not necessarily compact, which is the case occurring in the present paper, the above estimates have a local character and a suitable technique has to be developed to make them valid globally. Given a mild solution $x_0 \in \mathcal{M}_{A,F,a}$ and using the properties of the contraction semigroup $T(t)$, we first construct for the Cauchy problem $(C_{A,F,a})$ a local approximate mild solution y_η which is close to x_0 and has pseudoderivative $u_{y_\eta}(t)$ close to the extreme points of $F(t, y_\eta(t))$. This is achieved by means of an appropriate discretization technique of the semigroup $T(t)$, devised here for that purpose. Then, by virtue of a smooth version of the Filippov–Ważewski theorem, which is valid under the assumption that $F(t, x)$ is locally Lipschitzian in the x-variable (Theorem 3.4), we construct close to y_η and thus to x_0, an exact mild solution z with pseudoderivative $u_z(t)$ still close to the extreme points of $F(t, z(t))$. Finally a delicate Zorn type argument makes it possible to extend z as a global mild solution of $(C_{A,F,a})$, in such a way that z retains the above mentioned properties. Thus z is in $\mathcal{M}_n$ and is close to x_0, and hence $\mathcal{M}_n$ is dense in $\mathcal{M}_{A,F,a}$.

Our investigation has been confined to the case where $T(t)$ is contractive in order to avoid additional (not yet settled) technical difficulties which occur if the semigroup $T(t)$ is merely strongly continuous, or more generally, if its infinitesimal generator is time dependent. In the simpler case where $A = 0$ an analogous bang-bang result was proved in [14].

The present Baire category approach to the bang-bang property for the Cauchy problem $(C_{A,F,a})$ is essentially elementary though rather technical. Moreover it works without any compactness assumption on the semigroup $T(t)$ or the multifunction $F(t, x)$. In view of that, Theorem 5.1 appears to be new even under the stronger hypothesis that $F(t, x)$ is globally Lipschitzian in the x-variable.

It is not clear if Theorem 5.1 can be proved more directly by using the selection type approach developed by Tolstonogov [34–36]. In a setting similar to ours, such an approach has been employed by Wang [38], under the assumption that $F(t, x)$ is Caratheodory, globally Lipschitzian in the x-variable and, in addition, satisfies a suitable compactness condition. For some other classes of evolution differential inclusions similar bang-bang and relaxation properties have been investigated, by different techniques, by Donchev, Farkhi and Mordukhovich [16], Frankowska [18], Ingall, Sontag and Wang [22], Papageorgiou [26], Papageorgiou and Shahzad [28], Tolstonogov [37].

In conclusion, it is worthwhile to observe that the theory of evolution differential inclusions, besides its intrinsic interest as a possible source of new interesting problems, can be looked as an abstract and useful framework for some classes of distributed parameter control problems. In this context, exhaustive information on the general theory (including the set valued calculus) and on its applications can be found in the monographs of Hu and Papageorgiou [20] and Mordukhovich [23].

The present paper is divided into 5 sections, with the introduction. Section 2 contains notations and preliminaries. In Section 3 a Filippov–Ważewski type theorem is proved. Some technical approximation results are established in Section 4 and, in Section 5, they are used to prove the above mentioned bang-bang property (1.1).

2. Notations and preliminaries

Throughout the present paper $\mathbb{E}$ is a reflexive and separable real Banach space with norm $\|.\|$ and $\mathcal{C}(\mathbb{E})$ (resp. $\mathcal{K}(\mathbb{E})$) is the space of all nonempty closed convex bounded (resp. nonempty closed bounded) subsets of $\mathbb{E}$ endowed with the Pompeiu–Hausdorff metric

$$h(A,B)=\max\Big\{\sup_{a\in A}d(a,B),\ \sup_{b\in B}d(b,A)\Big\}.$$

Here, for $x\in\mathbb{E}$ and $\emptyset\neq Z\subset\mathbb{E}$, $d(x,Z)=\inf_{z\in Z}\|x-z\|$.

If $A\subset\mathbb{E}$ then $co\,A$ and $\overline{co}\,A$ denote the convex hull and the closed convex hull of A. If $A\subset\mathbb{E}$ is convex, $\operatorname{ext}A$ denotes the set of the extreme points of A.

In any metric space M an open and a closed ball with center $x\in M$ and radius $r>0$ are denoted by $B(x,r)$ and $B[x,r]$. If $A\subset M$, by $\overline{A}$ we mean the closure of A in M.

In the sequel, $I=[t_0,t_1]$, where $t_0<t_1$. If $J\subset\mathbb{R}$ the Lebesgue measure of J is denoted by $|J|$, while χ_J stands for the characteristic function of J.

The space $I\times\mathbb{E}$ is equipped with the metric

$$\max\big\{\big|t'-t''\big|,\big\|x'-x''\big\|\big\},\quad \big(t',x'\big),\big(t'',x''\big)\in I\times\mathbb{E}.$$

For any continuous function $z:I\to\mathbb{E}$ and $R>0$, the set

$$N(z,R)=\big\{(t,x)\in I\times\mathbb{E}\ \big|\ t\in I,\ \big\|x-z(t)\big\|<R\big\}$$

is called a *tube* around the graph of z.

Consider a multifunction $F:I\times\mathbb{E}\to\mathcal{K}(\mathbb{E})$. F is *locally Lipschitzian* in the x-variable if for each $(t,x)\in I\times\mathbb{E}$ there exist $\delta_{t,x}>0$ and a constant $K_{t,x}\geqslant 0$ such that $(s,u),(s,v)\in B((t,x),\delta_{t,x})$ implies $h(F(s,u),F(s,v))\leqslant K_{t,x}\|u-v\|$. F is K*-Lipschitzian* in the x-variable in the tube $N(z,R)$ if there exists a constant $K\geqslant 0$ such that $(s,u),(s,v)\in N(z,R)$ implies $h(F(s,u),F(s,v))\leqslant K\|u-v\|$.

Evidently the above constants $K_{t,x}$ and K can be assumed, without loss of generality, to be strictly positive.

By virtue of Lebesgue's covering lemma we have the following

Lemma 2.1. *Let $F:I\times\mathbb{E}\to\mathcal{K}(\mathbb{E})$ be locally Lipschitzian in the x-variable and let $z:I\to\mathbb{E}$ be continuous. Then there exist $R>0$ and $K>0$ such that F is K-Lipschitzian in the tube $N(z,R)$.*

The above numbers R and K are said, for brevity, corresponding to z.

We denote by $C(I,\mathbb{E})$ the Banach space of all continuous functions $x:I\to\mathbb{E}$ equipped with the norm of uniform convergence $\|x\|_I=\max\{\|x(t)\|:\ t\in I\}$. The meaning of $L^p(I,\mathbb{E})$, $1\leqslant p\leqslant\infty$, is the standard one.

For $a\in\mathbb{E}$ consider the Cauchy problems $(C_{A,F,a})$ and $(C_{A,\operatorname{ext}F,a})$.

On the operator A and on the multifunction $F:I\times\mathbb{E}\to\mathcal{C}(\mathbb{E})$ (or $F:I\times\mathbb{E}\to\mathcal{K}(\mathbb{E})$), where $I=[t_0,t_1]$, we make the following assumptions:

(h_1) A is the infinitesimal generator of a C_0-semigroup of contractions $T(t)$, $t\geqslant 0$, on $\mathbb{E}$,
(h_2) F is continuous on $I\times\mathbb{E}$,
(h_3) F is locally Lipschitzian in the x-variable,
(h_4) $\|F(t,x)\|\leqslant M$ for $(t,x)\in I\times\mathbb{E}$, where M is a positive constant.

Remark 2.1. If (h_1) holds then by renorming the space $\mathbb{E}$ (see Pazy [29, Chapter 1]) we can and do assume, without loss of generality, that each linear operator $T(t)$, $t\geqslant 0$, satisfies $\|T(t)\|\leqslant 1$ or equivalently

$$\big\|T(t)x\big\|\leqslant\|x\|\quad\text{for every } x\in\mathbb{E}.$$

In the following definitions we consider the Cauchy problems $(C_{A,F,a})$ and $(C_{A,\operatorname{ext}F,a})$ in which A and $F:I\times\mathbb{E}\to\mathcal{C}(\mathbb{E})$ are supposed to satisfy (h_1)–(h_4).

Definition 2.1. A function $x:I\to\mathbb{E}$ is said to be a mild solution of the Cauchy problem $(C_{A,F,a})$ (resp. $(C_{A,\operatorname{ext}F,a})$) if x is continuous on I and there exists a Bochner integrable function $u_x:I\to\mathbb{E}$ such that

$$x(t)=T(t-t_0)a+\int_{t_0}^{t}T(t-s)u_x(s)\,ds,\quad t\in I,$$

$$u_x(t)\in F\big(t,x(t)\big)\quad\big(\text{resp. } u_x(t)\in\operatorname{ext}F\big(t,x(t)\big)\big),\quad t\in I \text{ a.e.}$$

A mild solution $x:I\to\mathbb{E}$ of $(C_{A,F,a})$ such that $u_x:I\to\mathbb{E}$ is continuous is called smooth mild solution of $(C_{A,F,a})$.

For A and $F : I \times \mathbb{E} \to \mathcal{C}(\mathbb{E})$ satisfying (h_1)–(h_4) we set

$$\mathcal{M}_{A,F,a} = \{x \in C(I,\mathbb{E}):\ x \text{ is a mild solution of } (C_{A,F,a})\},$$
$$\mathcal{S}_{A,F,a} = \{x \in C(I,\mathbb{E}):\ x \text{ is a smooth mild solution of } (C_{A,F,a})\},$$
$$\mathcal{M}_{A,\operatorname{ext} F,a} = \{x \in C(I,\mathbb{E}):\ x \text{ is a mild solution of } (C_{A,\operatorname{ext} F,a})\}.$$

The space $\mathcal{M}_{A,F,a}$ is endowed with the induced metric of $C(I,\mathbb{E})$.

Definition 2.2. A function $y_\eta : I \to \mathbb{E}$ is said to be a mild η-solution (resp. smooth mild η-solution) of the Cauchy problem $(C_{A,F,a})$ if y_η is continuous on I and there exist a Bochner integrable (resp. continuous) function $v_{y_\eta} : I \to \mathbb{E}$ and $\eta > 0$ such that

$$y_\eta(t) = T(t-t_0)a + \int_{t_0}^{t} T(t-s)v_{y_\eta}(s)\,ds, \quad t \in I,$$
$$\int_I d\big(v_{y_\eta}(t), F\big(t, y_\eta(t)\big)\big)\,dt \leqslant \eta.$$

For $F : I \times \mathbb{E} \to \mathcal{K}(\mathbb{E})$ the definitions of mild solution and mild η-solution are unchanged.

Remark 2.2. The function $u_x : I \to \mathbb{E}$ associated to a mild solution $x \in C(I,\mathbb{E})$ according to Definition 2.1 is unique in the $L^\infty(I,\mathbb{E})$ sense (see [15]).

For brevity we say that u_x corresponds to x and we call u_x *pseudoderivative* of x. Remark 2.2 remains valid also for mild η-solutions.

In the sequel we use the notation $(C_{A,F,a,\tau})$ to denote a Cauchy problem with initial condition $x(\tau) = a$, where $t_0 \leqslant \tau < t_1$, i.e.

$$\dot{x}(t) \in Ax(t) + F\big(t, x(t)\big), \quad x(\tau) = a. \tag{$C_{A,F,a,\tau}$}$$

By virtue of Michael selection theorem, see [21], we have the following

Lemma 2.2. *Let $u : I \to \mathbb{E}$ and $G : I \to \mathcal{C}(\mathbb{E})$ be continuous. Then for every $\varepsilon > 0$ there exists a continuous function $v : I \to \mathbb{E}$ such that*

$$v(t) \in G(t) \cap B\big[u(t), d\big(u(t), G(t)\big) + \varepsilon\big], \quad t \in I.$$

The Choquet function which we now introduce plays a crucial role in the proof of our main result.

Denote by $\mathbb{E}^*$ the topological dual of $\mathbb{E}$. Let $\{l_n\}$, $\|l_n\| = 1$, be a sequence dense in the unit sphere of $\mathbb{E}^*$. Let F satisfy (h_2)–(h_4). Define $\varphi_F : I \times \mathbb{E} \times \mathbb{E} \to [0, +\infty]$ by

$$\varphi_F(t,x,v) = \begin{cases} \sum_{n=1}^{\infty} \frac{(l_n(v))^2}{2^n}, & v \in F(t,x), \\ +\infty, & v \in \mathbb{E} \setminus F(t,x). \end{cases}$$

Let $\mathcal{A}$ be the set of all continuous affine functions $a : \mathbb{E} \to \mathbb{R}$. Let $\overline{\varphi}_F : I \times \mathbb{E} \times \mathbb{E} \to [-\infty, +\infty)$ be given by

$$\overline{\varphi}_F(t,x,v) = \inf\{a(v) \mid a \in \mathcal{A} \text{ and } a(z) > \varphi_F(t,x,z) \text{ for every } z \in F(t,x)\}.$$

We define $d_F : I \times \mathbb{E} \times \mathbb{E} \to [-\infty, +\infty)$ by

$$d_F(t,x,v) = \overline{\varphi}_F(t,x,v) - \varphi_F(t,x,v).$$

In the next lemma we review some properties of d_F, the Choquet function associated to F (see Choquet [7], Castaing and Valadier [4]).

Lemma 2.3. *Let $F : I \times \mathbb{E} \to \mathcal{C}(\mathbb{E})$ satisfy (h_2)–(h_4). Then we have*:

(i) *For each $(t,x) \in I \times \mathbb{E}$ and $v \in F(t,x)$ we have $0 \leqslant d_F(t,x,v) \leqslant M^2$. Moreover $d_F(t,x,v) = 0$ if and only if $v \in \operatorname{ext} F(t,x)$.*
(ii) *For each $(t,x) \in I \times \mathbb{E}$, the function $d_F(t,x,.)$ is concave on $\mathbb{E}$ and strictly concave on $F(t,x)$.*
(iii) *d_F is upper semicontinuous on $I \times \mathbb{E} \times \mathbb{E}$.*

(iv) *For each $x \in \mathcal{M}_{A,F,a}$ with corresponding pseudoderivative $u_x \in L^\infty(I,\mathbb{E})$, the function $t \to d_F(t, x(t), u_x(t))$ is nonnegative, bounded and integrable on I.*

(v) *If $\{x_n\} \subset \mathcal{M}_{A,F,a}$ converges to $x \in \mathcal{M}_{A,F,a}$ and $\{u_{x_n}\}$ converges weakly in $L^1(I,\mathbb{E})$ to u_x we have*

$$\limsup_{n\to\infty} \int_I d_F\big(t, x_n(t), u_{x_n}(t)\big)\,dt \leqslant \int_I d_F\big(t, x(t), u_x(t)\big)\,dt.$$

3. Auxiliary results

In this section we prove some auxiliary results which will be useful in what follows.

Theorem 3.1. *Let A and $F : I \times \mathbb{E} \to \mathcal{K}(\mathbb{E})$ satisfy (h_1)–(h_4). Then, for each $a \in \mathbb{E}$ and $t_0 < \tau < t_1$, the Cauchy problem $(C_{A,F,a,\tau})$ has a local mild solution $x : J \to \mathbb{E}$ defined in some interval $J = [\tau, \tau'] \subset I$.*

The proof is similar with that of Filippov [17] (see also [1]) and is omitted.

Theorem 3.2. *Let A and $F : I \times \mathbb{E} \to \mathcal{K}(\mathbb{E})$ satisfy (h_1)–(h_4). Then, for each $a \in \mathbb{E}$, the Cauchy problem $(C_{A,F,a})$ has a mild solution $x : I \to \mathbb{E}$.*

Proof. Without loss of generality we assume $M > 1$, where M is the constant in (h_4). Denote by Λ the set of all mild solutions $x : [t_0, \tau] \to \mathbb{E}$ of $(C_{A,F,a})$ defined on $[t_0, \tau]$ with $t_0 < \tau \leqslant t_1$. By Theorem 3.1, Λ is nonempty.

For $x_\alpha, x_\beta \in \Lambda$ where $x_\alpha : [t_0, \tau_\alpha] \to \mathbb{E}$ and $x_\beta : [t_0, \tau_\beta] \to \mathbb{E}$, we define

$$x_\alpha \prec x_\beta \quad \text{if and only if} \quad \tau_{x_\alpha} \leqslant \tau_{x_\beta} \quad \text{and} \quad x_\alpha(t) = x_\beta(t), \quad t \in [t_0, \tau_{x_\alpha}].$$

It is easy to see that $(\Lambda, \prec)$ is a partially ordered set.

Claim 1. *Every totally ordered subset $\Gamma = \{x_j\}_{j\in J}$ of Λ has an upper bound, i.e. there exists $x^* \in \Lambda$ such that $x_j \prec x^*$ for every $j \in J$.*

Let $\tau^* = \sup\{\tau_j\colon\ j \in J\}$. Suppose $\tau_{x_j} < \tau^*$ for every $j \in J$ (if $\tau_{x_{j_0}} = \tau^*$ for some $j_0 \in J$ there is nothing to prove for x_{j_0} would be an upper bound of Γ). Let $\{\tau_{x_{j_n}}\}$ be a strictly increasing sequence converging to τ^* and let $u_{x_{j_n}} : [t_0, \tau_{x_{j_n}}]$ correspond to $x_{x_{j_n}} : [t_0, \tau_{x_{j_n}}] \to \mathbb{E}$, according to Remark 2.2. For brevity we write τ_n, x_n, u_n instead of $\tau_{x_{j_n}}$, x_{j_n}, $u_{x_{j_n}}$. For $n < m$ we have $\tau_n < \tau_m$ and thus $x_n \prec x_m$ which implies $x_n(t) = x_m(t)$, $t \in [t_0, \tau_n]$ and $u_n(t) = u_m(t)$, $t \in [t_0, \tau_n]$ a.e. Define $x^* : [t_0, \tau^*] \to \mathbb{E}$ and $u^* : [t_0, \tau^*] \to \mathbb{E}$ by

$$x^*(t) = x_n(t), \quad t \in [t_0, \tau_n], \qquad u^*(t) = u_n(t), \quad t \in [t_0, \tau_n] \text{ a.e.}$$

It is evident that the functions x^* and u^* are well defined and that, on $[t_0, \tau^*)$, x^* is continuous and u^* is Bochner integrable. Furthermore $x^* : [t_0, \tau^*) \to \mathbb{E}$ (with pseudoderivative $u^* : [t_0, \tau^*) \to \mathbb{E}$) is a mild solution of $(C_{A,F,a})$ defined on $[t_0, \tau^*)$ because, for each $n \in \mathbb{N}$, $x_n : [t_0, \tau^*) \to \mathbb{E}$ (with pseudoderivative $u_n : [t_0, \tau^*) \to \mathbb{E}$) is a mild solution of $(C_{A,F,a})$ defined on $[t_0, \tau_n]$ and $\tau_n \to \tau^*$. The proof of Claim 1 is achieved if we show that x^* can be extended by continuity all over $[t_0, \tau^*]$, i.e. if we prove the following

Claim 2. *There exists $\xi \in \mathbb{E}$ such that*

$$\lim\nolimits_{t\to\tau^*} x^*(t) = \xi. \tag{3.1}$$

For this it suffices to show that x^* is uniformly continuous on $[t_0, \tau^*)$. Let $\varepsilon > 0$. For arbitrary $t, t' \in [t_0, \tau^*)$, $t < t'$, setting $t' = t + h$ we have

$$\begin{aligned}
x^*\big(t'\big) - x^*(t) &= T\big(t' - t_0\big)a + \int_{t_0}^{t'} T\big(t' - s\big)u^*(s)\,ds - T(t - t_0)a - \int_{t_0}^{t} T(t - s)u^*(s)\,ds \\
&= T(t - t_0)\big(T(h)a - a\big) + \int_{t_0}^{t_0+h} T(t + h - s)u^*(s)\,ds \\
&\quad + \left(\int_{t_0+h}^{t+h} T(t + h - s)u^*(s)\,ds - \int_{t_0}^{t} T(t - s)u^*(s)\,ds\right).
\end{aligned}$$

Hence

$$\left\|x^*\left(t'\right)-x^*(t)\right\| \leqslant \left\|T(h)a-a\right\|+hM+\left\|\int_{t_0}^{t} T(t-s)u^*(s+h)\,ds-\int_{t_0}^{t} T(t-s)u^*(s)\,ds\right\|,$$

and so

$$\left\|x^*\left(t'\right)-x^*(t)\right\| \leqslant \left\|T(h)a-a\right\|+hM+\int_{t_0}^{t} \left\|u^*(s+h)-u^*(s)\,ds\right\|. \tag{3.2}$$

Let $g:[t_0,\tau^*]\to\mathbb{E}$ be a continuous function satisfying

$$\int_{t_0}^{\tau^*} \left\|g(s)-u^*(s)\right\|ds<\frac{\varepsilon}{8}.$$

Since

$$\begin{aligned}\int_{t_0}^{t} \left\|u^*(s+h)-u^*(s)\right\|ds &\leqslant \int_{t_0}^{t} \left\|u^*(s+h)-g(s+h)\right\|ds+\int_{t_0}^{t} \left\|g(s+h)-g(s)\right\|ds\\ &\quad+\int_{t_0}^{t} \left\|g(s)-u^*(s)\right\|ds<\frac{\varepsilon}{4}+\int_{t_0}^{t} \left\|g(s+h)-g(s)\right\|ds,\end{aligned}$$

then from (3.2) we have

$$\left\|x^*\left(t'\right)-x^*(t)\right\| < \left\|T(h)a-a\right\|+hM+\frac{\varepsilon}{4}+\int_{t_0}^{t} \left\|g(s+h)-g(s)\right\|ds. \tag{3.3}$$

Now $\|T(h)a-a\|\to 0$, for $h\to 0$, and g is uniformly continuous on $[t_0,\tau^*]$, hence there exists a δ, $0<\delta<\frac{\varepsilon}{4M}$, such that $0<h<\delta$ implies

$$\left\|T(h)a-a\right\|<\frac{\varepsilon}{4},\quad hM<\varepsilon/4,\qquad \left\|g(s+h)-g(s)\right\|<\frac{\varepsilon}{4(\tau^*-t_0)},\quad s\in[t_0,t].$$

From the latter and (3.3) it follows that $\|x^*(t')-x^*(t)\|<\varepsilon$, for all $t,t'\in[t_0,\tau^*)$ with $|t'-t|<\delta$. Therefore x^* is uniformly continuous on $[t_0,\tau^*)$ and Claim 2 is proved.

In view of (3.1), the mild solution x^* defined on $[t_0,\tau^*)$ has a continuous extension, say $x^*:[t_0,\tau^*]\to\mathbb{E}$, defined on the closed interval $[t_0,\tau^*]$. Obviously $x^*:[t_0,\tau^*]\to\mathbb{E}$ is a mild solution of $(C_{A,F,a})$ and thus $x^*\in\Lambda$. Moreover, $x_n\prec x^*$ for each $n\in\mathbb{N}$ since $\tau_n<\tau^*$ and $x_n(t)=x^*(t)$, $t\in[t_0,\tau_n]$. Since $\tau_n\to\tau^*$, for each $x_j\in\Gamma$ there exists $n\in\mathbb{N}$ such that $x_j\prec x_n$. Hence $x_j\prec x^*$ for every $j\in J$, i.e. $x^*\in\Lambda$ is an upper bound of Γ and Claim 1 holds.

By Zorn's lemma the set Λ has a maximal element, say $x:[t_0,\tau]\to\mathbb{E}$, defined on the closed interval $[t_0,\tau]$, for some $\tau\in(t_0,t_1]$. If $\tau<t_1$ then by Theorem 3.1 one can construct a mild solution of $(C_{A,F,a})$, say $\tilde{x}:[t_0,\tau+\delta]\to\mathbb{E}$, for some $\delta>0$, such that $x\prec\tilde{x}$, $x\neq\tilde{x}$, a contradiction to the maximality of x. Consequently $\tau=t_1$, and thus the Cauchy problem $(C_{A,F,a})$ has a mild solution $x:I\to\mathbb{E}$ defined all over I. This completes the proof. □

Theorem 3.3. *Let A and $F:I\times\mathbb{E}\to\mathcal{C}(\mathbb{E})$ satisfy (h_1)–(h_4). Then $\mathcal{M}_{A,F,a}$ is a nonempty closed subset of $C(I,\mathbb{E})$. Moreover $\mathcal{M}_{A,F,a}$ is a nonempty complete metric space under the induced metric of $C(I,\mathbb{E})$.*

Proof. By Theorem 3.2 $\mathcal{M}_{A,F,a}$ is nonempty. As F is a continuous and bounded multifunction with closed convex values contained in $\mathbb{E}$, a reflexive Banach space, the uniform limit of mild solutions of $(C_{A,F,a})$ is a mild solution of $(C_{A,F,a})$. Therefore $(\mathcal{M}_{A,F,a})$ is closed in $C(I,\mathbb{E})$. The last statement is obvious. □

The next theorem is a smooth version of the Filippov–Ważewski theorem concerning smooth mild solutions to the Cauchy problem $(C_{A,F,a})$, with F a locally Lipschitzian convex-valued multifunction.

Theorem 3.4. *Let A and $F:I\times\mathbb{E}\to\mathcal{C}(\mathbb{E})$ satisfy (h_1)–(h_4). Let $z:I\to\mathbb{E}$ be a mild solution of $(C_{A,F,a})$, with corresponding $R=R_z>0$ and $K=K_z>0$, and let $\xi_0\in B(z(t_0),R/3)$. For any η, with $0<\eta<\frac{R}{9}e^{-K|I|}$, let $y_\eta:I\to\mathbb{E}$ be a smooth mild η-solution of (C_{A,F,ξ_0}), i.e.*

$$y_\eta(t) = T(t-t_0)\xi_0 + \int_{t_0}^{t} T(t-s)v_{y_\eta}(s)\,ds, \quad t \in I,$$

$$\int_{t_0}^{t_1} p(t)\,dt \leqslant \eta \quad \text{where } p(t) = d\big(v_{y_\eta}(t), F\big(t, y_\eta(t)\big)\big),$$

and $v_{y_\eta} : I \to \mathbb{E}$, corresponding to y_η, is continuous. Suppose further,

$$\big\|y_\eta(t) - z(t)\big\| < \frac{R}{3}, \quad t \in I.$$

Then, for every $0 < \varepsilon < R/9$ and $\xi \in \mathbb{E}$, with $\|\xi - \xi_0\| < \frac{R}{9}e^{-K|I|}$, there exists a smooth mild solution $x : I \to \mathbb{E}$ of $(C_{A,F,\xi})$ with the following properties:

$$\big\|x(t) - z(t)\big\| < \frac{2}{3}R, \quad t \in I, \tag{3.4}$$

$$\big\|x(t) - y_\eta(t)\big\| \leqslant q(t) + \varepsilon(t-t_0), \quad t \in I, \tag{3.5}$$

$$\big\|u_x(t) - v_{y_\eta}(t)\big\| \leqslant Kq(t) + p(t) + \varepsilon, \quad t \in I, \tag{3.6}$$

where $u_x : I \to \mathbb{E}$, corresponding to x, is continuous and

$$q(t) = \big\|\xi - \xi_0\big\|e^{K(t-t_0)} + \int_{t_0}^{t} e^{K(t-s)}p(s)\,ds, \quad t \in I.$$

Proof. Without loss of generality we assume $K > 1$. As in Filippov [17] we shall construct a suitable sequence $\{x_n\}$ of smooth mild ε-solutions of $(C_{A,F,\xi})$ given by

$$x_n(t) = T(t-t_0)\xi + \int_{t_0}^{t} T(t-s)u_n(s)\,ds, \quad t \in I,$$

such that $\{x_n\}$ and $\{u_n\}$ are both Cauchy sequences in $C(I, \mathbb{E})$.

Let $0 < \varepsilon < R/9$ be arbitrary, and set $x_0(t) = y_\eta(t)$, $u_0(t) = v_{y_\eta}(t)$, $t \in I$, $\varepsilon_{n-1} = \varepsilon/e^{K|I|}2^n$, $n \in \mathbb{N}$. By Lemma 2.2 there exists $u_1 \in C(I, \mathbb{E})$ such that

$$u_1(t) \in F\big(t, x_0(t)\big) \cap B\big[u_0(t), d\big(u_0(t), F\big(t, x_0(t)\big)\big) + \varepsilon_0\big], \quad t \in I. \tag{3.7}$$

Define $x_1 : I \to \mathbb{E}$ by

$$x_1(t) = T(t-t_0)\xi + \int_{t_0}^{t} T(t-s)u_1(s)\,ds, \quad t \in I.$$

Clearly $x_1 \in C(I, \mathbb{E})$. Set $\delta = \|\xi - \xi_0\|$. In view of (3.7) we have

$$\begin{aligned}\big\|x_1(t) - x_0(t)\big\| &\leqslant \big\|T(t-t_0)(\xi - \xi_0)\big\| + \int_{t_0}^{t} \big\|T(t-s)\big\|\big\|u_1(s) - u_0(s)\big\|\,ds \\ &\leqslant \delta + \int_{t_0}^{t} p(s)\,ds + \varepsilon_0(t-t_0), \quad t \in I.\end{aligned}$$

Since $\int_{t_0}^{t} p(s)\,ds \leqslant \eta$ and δ, η, $\varepsilon_0|I|$ are smaller than $R/9$, it follows

$$\big\|x_1(t) - x_0(t)\big\| \leqslant \delta + \eta + \varepsilon_0|I| < R/3, \quad t \in I.$$

We now construct by induction two sequences $\{u_n\}, \{x_n\} \subset C(I, \mathbb{E})$ satisfying, for every $n \in \mathbb{N}$ and every $t \in I$, the following properties:

$$u_n(t) \in F\big(t, x_{n-1}(t)\big) \cap B\big[u_{n-1}(t), d\big(u_{n-1}(t), F\big(t, x_{n-1}(t)\big)\big) + \varepsilon_{n-1}\big], \tag*{(3.8)$_n$}$$

$$x_n(t) = T(t-t_0)\xi + \int_{t_0}^{t} T(t-s)u_n(s)\,ds, \tag{3.9$_n$}$$

$$\|x_n(t) - x_{n-1}(t)\| \leqslant \delta \frac{[K(t-t_0)]^{n-1}}{(n-1)!} + \int_{t_0}^{t} \frac{[K(t-s)]^{n-1}}{(n-1)!} p(s)\,ds + \frac{1}{K}\sum_{i=0}^{n-1} \varepsilon_i \frac{[K(t-t_0)]^{n-i}}{(n-i)!}, \tag{3.10$_n$}$$

$$\|x_n(t) - x_0(t)\| \leqslant \delta \sum_{r=1}^{n} \frac{(K|I|)^{r-1}}{(r-1)!} + \eta \sum_{r=1}^{n} \frac{(K|I|)^{r-1}}{(r-1)!} + \frac{1}{K}\sum_{r=1}^{n}\sum_{i=0}^{r-1} \varepsilon_i \frac{(K|I|)^{r-i}}{(r-i)!}. \tag{3.11$_n$}$$

It has been shown that $u_1, x_1 \in C(I,\mathbb{E})$ satisfy $(3.8)_1$–$(3.11)_1$. Then, as in [14], one can show that $(3.8)_n$–$(3.11)_n$ are valid for every $n \in \mathbb{N}$ and that the sequences $\{u_n\}$, $\{x_n\}$ converge respectively to $u, x \in C(I,\mathbb{E})$. From $(3.8)_n$ and $(3.9)_n$ letting $n \to \infty$ it follows that x is a mild solution of $(C_{A,F,\xi})$, with pseudoderivative $u_x = u$. The proof that x and u_x satisfy (3.4)–(3.6) is as in [14] and thus it is omitted. □

Theorem 3.5. *Let A and $F : I \times \mathbb{E} \to \mathcal{C}(\mathbb{E})$ satisfy (h_1)–(h_4). Then $\mathcal{S}_{A,F,a}$ is dense in $\mathcal{M}_{A,F,a}$.*

Proof. Let $z \in \mathcal{M}_{A,F,a}$ and let $R = R_z > 0$, $K = K_z > 0$ correspond to z. Since z is a mild solution of $(C_{A,F,a})$ there exists a Bochner integrable function $u_z : I \to \mathbb{E}$ such that

$$z(t) = T(t-t_0)a + \int_{t_0}^{t} T(t-s)u_z(s)\,ds, \quad t \in I,$$

$$u_z(t) \in F\big(t, z(t)\big), \quad t \in I \text{ a.e.} \tag{3.12}$$

Without loss of generality we assume $K > 1$ and $M > 1$ (M the constant in (h_4)).

Let $\varepsilon > 0$. Fix σ and η as follows

$$0 < \sigma < \min\left\{\frac{R}{9}, \frac{\varepsilon}{2(|I|+1)}\right\}, \qquad 0 < \eta < \frac{\sigma}{e^{K|I|}}. \tag{3.13}$$

By Lusin's theorem there exists a closed set $C \subset I$, with

$$|I \setminus C| < \frac{\eta}{2KM(|I|+1)}, \tag{3.14}$$

so that u_z restricted to C is continuous and satisfies (3.12) for all $t \in C$. Clearly $\|u_z(t)\| \leqslant M$, $t \in C$. Hence by Dugundji's theorem u_z admits a continuous extension, say v, all over I such that $\|v(t)\| \leqslant M$, $t \in I$.

Define $y : I \to \mathbb{E}$ by

$$y(t) = T(t-t_0)a + \int_{t_0}^{t} T(t-s)v(s)\,ds, \quad t \in I.$$

We now prove that $y \in C(I,\mathbb{E})$ is a smooth mild η-solution of $(C_{A,F,a})$. To this end it suffices to show that

$$\int_I p(t)\,dt \leqslant \eta \quad \text{where } p(t) = d\big(v(t), F\big(t, y(t)\big)\big). \tag{3.15}$$

Clearly,

$$\|y(t) - z(t)\| \leqslant \int_{t_0}^{t} \|T(t-s)\| \|v(s) - u_z(s)\|\,ds \leqslant 2M|I \setminus C|, \quad t \in I. \tag{3.16}$$

In view of (3.14) and (3.13), $2KM(|I|+1)|I \setminus C| < \eta < \sigma$, and so we have

$$\|y(t) - z(t)\| < \sigma < \varepsilon/2, \quad t \in I. \tag{3.17}$$

Moreover,

$$d\big(v(t), F\big(t, y(t)\big)\big) \leqslant \|v(t) - u_z(t)\| + d\big(u_z(t), F\big(t, z(t)\big)\big) + h\big(F\big(t, z(t)\big), F\big(t, y(t)\big)\big), \quad t \in I,$$

and thus

$$d\big(v(t), F\big(t, y(t)\big)\big) \leqslant h\big(F\big(t, z(t)\big), F\big(t, y(t)\big)\big), \quad t \in C,$$

since for $t \in C$ we have $v(t) = u_z(t)$ and $u_z(t) \in F(t, z(t))$. Furthermore $\|y(t) - z(t)\| < \sigma < R/9$, $t \in I$, by virtue of (3.17) and (3.13). As F is K-Lipschitzian in the x-variable in the tube $N(z, R)$, in view of (3.16) it follows that

$$d\big(v(t), F\big(t, y(t)\big)\big) \leqslant K\big\|y(t) - z(t)\big\| \leqslant 2KM|I \setminus C|, \quad t \in C. \tag{3.18}$$

On the other hand, for $t \in I \setminus C$, $v(t)$ and $F(t, y(t))$ are bounded by M. In view of that and of (3.18), we have

$$\begin{aligned}\int_I d\big(v(t), F\big(t, y(t)\big)\big)\, dt &= \int_C d\big(v(t), F\big(t, y(t)\big)\big)\, dt + \int_{I\setminus C} d\big(v(t), F\big(t, y(t)\big)\big)\, dt \\ &\leqslant 2KM|I||I \setminus C| + 2M|I \setminus C| < \eta,\end{aligned}$$

by virtue of (3.14). Therefore (3.15) holds and thus y is a smooth η-solution of $(C_{A,F,a})$.

In order to apply Theorem 3.4 observe that $0 < \sigma < R/9$ and $0 < \eta < \frac{R}{9}e^{-K|I|}$, by (3.13), and $\|y(t) - z(t)\| < R/9$, $t \in I$. Then, by Theorem 3.4 (with $\xi = \xi_0 = z(t_0) = a$ and y, v, σ in the place of $y_\eta, v_\eta, \varepsilon$) there exists a smooth mild solution $x : I \to \mathbb{E}$ of $(C_{A,F,a})$ satisfying

$$\big\|x(t) - y(t)\big\| \leqslant q(t) + \sigma(t - t_0), \quad t \in I, \tag{3.19}$$

where $q(t) = \int_{t_0}^{t} e^{K(t-s)} p(s)\, ds$, $t \in I$.

By virtue of (3.15) and (3.13) for every $t \in I$ we have $q(t) + \sigma(t - t_0) < \eta e^{K|I|} + \sigma|I| < \sigma(1 + |I|) < \varepsilon/2$. Then (3.19) implies

$$\big\|x(t) - y(t)\big\| < \varepsilon/2, \quad t \in I.$$

Combining the latter with (3.17) gives $\|x(t) - z(t)\| < \varepsilon$ for every $t \in I$. Since $z \in \mathcal{M}_{A,F,a}$ and $\varepsilon > 0$ are arbitrary and $x \in \mathcal{S}_{A,F,a}$, the set $\mathcal{S}_{A,F,a}$ is dense in $\mathcal{M}_{A,F,a}$. This completes the proof. □

4. Approximation theorems

In this section we establish some technical approximation results which will play a crucial role in the proof of the bang-bang property.

Lemma 4.1. *Let A and $F : I \times \mathbb{E} \to \mathcal{C}(\mathbb{E})$ satisfy (h_1)–(h_4). Let $z : I \to \mathbb{E}$ be a smooth mild solution of $(C_{A,F,a})$, with pseudoderivative $u_z : I \to \mathbb{E}$, and let $R = R_z > 0$, $K = K_z > 0$ correspond to z. Let $\varepsilon > 0$ and $\alpha > 0$ be given. For $\tau \in I$ and $\xi_0 \in B(z(\tau), r)$, where $0 < r < R/3$, denote by $\tilde{z} : [\tau, \theta] \to \mathbb{E}$ a smooth mild solution of $(C_{A,F,\xi_0,\tau})$, with pseudoderivative $u_{\tilde{z}} \in C([\tau, \theta], \mathbb{E})$, defined on some interval $[\tau, \theta]$, $\tau < \theta \leqslant t_1$, satisfying*

$$\big\|\tilde{z}(t) - z(t)\big\| < r, \quad t \in [\tau, \theta]. \tag{4.1}$$

Then, for some $\tau' \in (\tau, \theta]$, there exists a smooth mild solution $x : [\tau, \tau'] \to \mathbb{E}$ of $(C_{A,F,\xi_0,\tau})$, with pseudoderivative $u_x \in C([\tau, \tau'], \mathbb{E})$, such that

(i) $$x(t) = T(t - \tau)\xi_0 + \int_\tau^t T(t - s)u_x(s)\, ds, \quad t \in \big[\tau, \tau'\big],$$

$$u_x(t) \in F\big(t, x(t)\big), \quad t \in \big[\tau, \tau'\big],$$

(ii) $$\big\|x(t) - \tilde{z}(t)\big\| < \frac{2}{3}R, \quad t \in \big[\tau, \tau'\big],$$

(iii) $$\big\|x\big(\tau'\big) - \tilde{z}\big(\tau'\big)\big\| < \varepsilon\big(\tau' - \tau\big),$$

(iv) $$\big\|x(t) - \tilde{z}(t)\big\| < \varepsilon, \quad t \in \big[\tau, \tau'\big],$$

(v) $$\int_\tau^{\tau'} d_F\big(t, x(t), u_x(t)\big)\, dt < \alpha\big(\tau' - \tau\big).$$

Proof. Without loss of generality we assume

$$K > 1, \qquad M > 1, \qquad 0 < \varepsilon < \min\{1, R\}, \tag{4.2}$$

where M is the constant in (h_4). By hypothesis $\tilde{z}$ and $u_{\tilde{z}}$ are continuous on $[\tau, \theta]$ and satisfy

$$\tilde{z}(t) = T(t-\tau)\xi_0 + \int_\tau^t T(t-s)u_{\tilde{z}}(s)\,ds, \quad t \in [\tau, \theta],$$
$$u_{\tilde{z}}(t) \in F\big(t, \tilde{z}(t)\big), \quad t \in [\tau, \theta]. \tag{4.3}$$

As $u_{\tilde{z}}(\tau) \in F(\tau, \tilde{z}(\tau))$ there exist points $e_1^\tau, \dots, e_n^\tau \in \operatorname{ext} F(\tau, \tilde{z}(\tau))$ and strictly positive constants $\lambda_1, \dots, \lambda_n$, with $\lambda_1 + \dots + \lambda_n = 1$, such that

$$\left\| u_{\tilde{z}}(\tau) - \sum_{i=0}^n \lambda_i e_i^\tau \right\| < \varepsilon/4. \tag{4.4}$$

By Michael's theorem, for each $i = 1, \dots, n$, there exists a continuous selection $e_i : [\tau, \theta] \to \mathbb{E}$ of the multifunction $F(t, \tilde{z}(t))$, $t \in [\tau, \theta]$, such that $e_i(\tau) = e_i^\tau$. By Lemma 2.3, d_F is upper semicontinuous and is zero at $(\tau, \tilde{z}(\tau), e_i^\tau)$, $i = 1, \dots, n$, hence there exists δ, $0 < \delta < \varepsilon/12$, such that, for $i = 1, \dots, n$, if $(t, x) \in I \times \mathbb{E}$ and $v \in F(t, x)$ satisfy

$$t \in [\tau, \tau + \delta], \quad \|x - \tilde{z}(\tau)\| < \delta, \qquad \|v - e_i^\tau\| < \delta,$$

then we have

$$0 \leqslant d_F(t, x, v) < \frac{\alpha}{2}. \tag{4.5}$$

Take σ as follows

$$0 < \sigma < \min\left\{\frac{\varepsilon}{24M}, \frac{\delta}{4}, \alpha, \frac{R}{3} - r\right\}. \tag{4.6}$$

Fix now $\tau' \in (\tau, \theta)$, with

$$0 < \tau' - \tau < \frac{\sigma}{2KM} e^{-K|I|}, \tag{4.7}$$

sufficiently close to τ so that the following properties are satisfied

$$\|T(t-s)e_i^\tau - e_i^\tau\| < \sigma, \quad \text{if } t \in [\tau, \tau'],\ s \in [\tau, t],\ i = 1, \dots, n, \tag{4.8}$$
$$\|T(t-\tau)\tilde{z}(\tau) - \tilde{z}(\tau)\| < \sigma, \quad \text{if } t \in [\tau, \tau'], \tag{4.9}$$
$$\|T(t-s)u_{\tilde{z}}(\tau) - u_{\tilde{z}}(\tau)\| < \sigma, \quad \text{if } t \in [\tau, \tau'],\ s \in [\tau, t], \tag{4.10}$$
$$\|u_{\tilde{z}}(t) - u_{\tilde{z}}(\tau)\| < \sigma, \qquad \|e_i(t) - e_i(\tau)\| < \sigma, \quad \text{if } t \in [\tau, \tau'],\ i = 1, \dots, n. \tag{4.11}$$

Such a τ' certainly exists as the maps $t \to T(t)v$, where $v \in \mathbb{E}$, $t \to u_{\tilde{z}}(t)$ and $t \to e_i(t)$ are continuous.

Divide $[\tau, \tau']$ into n intervals $J_i = [\tau_{i-1}, \tau_i]$, $i = 1, \dots, n$, where $\tau_0 = \tau$ and $\tau_i = \tau_{i-1} + \lambda_i(\tau' - \tau)$. Denote by $\Delta_i = (\delta_i, \eta_i)$, $i = 1, \dots, n-1$, a symmetric interval with center τ_i contained in $[\tau, \tau']$ and suppose that the intervals Δ_i are pairwise disjoint and satisfy

$$\sum_{i=1}^{n-1} |\Delta_i| < \frac{\sigma}{2M^2}\big(\tau' - \tau\big). \tag{4.12}$$

Moreover set

$$J_1' = [\tau_0, \delta_1], \quad J_2' = [\eta_1, \delta_2], \quad \dots, \quad J_{n-1}' = [\eta_{n-2}, \delta_{n-1}], \quad J_n' = [\eta_{n-1}, \tau_n].$$

For $i = 1, \dots, n-1$, define $\sigma_{\Delta_i} : \Delta_i \to \mathbb{E}$ by

$$\sigma_{\Delta_i}(t) = \frac{\eta_i - t}{|\Delta_i|} e_i(t) + \frac{t - \delta_i}{|\Delta_i|} e_{i+1}(t), \quad t \in \Delta_i,$$

and observe that σ_{Δ_i} is a continuous selection of the multifunction $F(t, \tilde{z}(t))$, $t \in \Delta_i$. Define now $\omega : [\tau, \tau'] \to \mathbb{E}$ and $y_\eta : [\tau, \tau'] \to \mathbb{E}$ as follows

$$\omega(t)=\sum_{i=1}^{n} e_i(t)\chi_{J_i'}(t)+\sum_{i=1}^{n-1}\sigma_{\Delta_i}(t)\chi_{\Delta_i}(t),\quad t\in[\tau,\tau'],$$

$$y_\eta(t)=T(t-\tau)\xi_0+\int_\tau^t T(t-s)\omega(s)\,ds,\quad t\in[\tau,\tau']. \tag{4.13}$$

Claim 1. *$y_\eta:[\tau,\tau']\to\mathbb{E}$ is a smooth mild η-solution of $(C_{A,F,\xi_0,\tau})$, with pseudoderivative $\omega\in C([\tau,\tau'],\mathbb{E})$, and $\eta=2KM(\tau'-\tau)^2$.*

It is evident that y_η and ω are continuous on $[\tau,\tau']$ and

$$\omega(t)\in F(t,\tilde z_\eta(t)),\quad t\in[\tau,\tau']. \tag{4.14}$$

Let us evaluate $p(t)=d(\omega(t),F(t,y_\eta(t)))$. Indeed, by virtue of (4.7), (4.2) and (4.6), one has

$$\|y_\eta(t)-\tilde z(t)\|\leqslant\int_\tau^t\|T(t-s)\|\big(\|\omega(s)\|+\|u_{\tilde z}(s)\|\big)\,ds\leqslant 2M(\tau'-\tau)<\frac{R}{3}-r,\quad t\in[\tau,\tau'], \tag{4.15}$$

and thus by (4.1),

$$\|y_\eta(t)-z(t)\|<\frac{R}{3},\quad t\in[\tau,\tau']. \tag{4.16}$$

In view of (4.14) and (4.15) it follows

$$\begin{aligned}p(t)&=d\big(\omega(t),F(t,y_\eta(t))\big)\leqslant d\big(\omega(t),F(t,\tilde z(t))\big)+h\big(F(t,\tilde z(t)),F(t,y_\eta(t))\big)\\&\leqslant K\|\tilde z(t)-y_\eta(t)\|\leqslant 2KM(\tau'-\tau),\quad t\in[\tau,\tau'],\end{aligned} \tag{4.17}$$

which implies

$$\int_\tau^{\tau'}p(t)\,dt\leqslant 2KM(\tau'-\tau)^2=\eta,$$

completing the proof of Claim 1.

Observe that

$$q(t)=\int_\tau^t e^{K(t-s)}p(s)\,ds\leqslant 2KM(\tau'-\tau)^2e^{K(\tau'-\tau)}<\sigma(\tau'-\tau),\quad t\in[\tau,\tau'], \tag{4.18}$$

for $\tau'-\tau<\sigma/(2KMe^{K|I|})$ by (4.7).

Claim 2. $\|y_\eta(\tau')-\tilde z(\tau')\|<\frac{\varepsilon}{2}(\tau'-\tau)$.

In fact, by (4.13) and (4.3), we have

$$\begin{aligned}y_\eta(\tau')-\tilde z(\tau')&=\int_\tau^{\tau'}T(\tau'-s)\omega(s)\,ds-\int_\tau^{\tau'}T(\tau'-s)u_{\tilde z}(s)\,ds\\&=\int_\tau^{\tau'}T(\tau'-s)\left(\sum_{i=1}^{n}e_i(s)\chi_{J_i'}(s)+\sum_{i=1}^{n-1}\sigma_{\Delta_i}(s)\chi_{\Delta_i}(s)\right)ds\\&\quad-\int_\tau^{\tau'}T(\tau'-s)u_{\tilde z}(\tau)\,ds+\int_\tau^{\tau'}T(\tau'-s)\big(u_{\tilde z}(\tau)-u_{\tilde z}(s)\big)\,ds\\&=\sum_{i=1}^{n}\int_{J_i'}T(\tau'-s)e_i(s)\,ds+\sum_{i=1}^{n-1}\int_{\Delta_i}T(\tau'-s)\sigma_{\Delta_i}(s)\,ds\end{aligned}$$

$$
\begin{aligned}
&- \left(\sum_{i=1}^{n} \int\limits_{J_i'} T(\tau' - s) u_{\tilde z}(\tau)\, ds + \sum_{i=1}^{n-1} \int\limits_{\Delta_i} T(\tau' - s) u_{\tilde z}(\tau)\, ds \right) + \int\limits_{\tau}^{\tau'} T(\tau' - s)\big(u_{\tilde z}(\tau) - u_{\tilde z}(s)\big)\, ds \\
&= \sum_{i=1}^{n} \left(\int\limits_{J_i'} e_i^\tau\, ds + \int\limits_{J_i'} \big(T(\tau' - s) e_i^\tau - e_i^\tau\big)\, ds + \int\limits_{J_i'} T(\tau' - s)\big(e_i(s) - e_i^\tau\big)\, ds \right) \\
&\quad - \sum_{i=1}^{n} \left(\int\limits_{J_i'} u_{\tilde z}(\tau)\, ds + \int\limits_{J_i'} \big(T(\tau' - s) u_{\tilde z}(\tau) - u_{\tilde z}(\tau)\big)\, ds \right) \\
&\quad + \sum_{i=1}^{n-1} \int\limits_{\Delta_i} T(\tau' - s)\big(\sigma_{\Delta_i}(s) - u_{\tilde z}(\tau)\big)\, ds + \int\limits_{\tau}^{\tau'} T(\tau' - s)\big(u_{\tilde z}(\tau) - u_{\tilde z}(s)\big)\, ds.
\end{aligned}
$$

Hence

$$
y_\eta(\tau') - \tilde z(\tau') = \sum_{i=1}^{n} \int\limits_{J_i} \big(e_i^\tau - u_{\tilde z}(\tau)\big)\, ds + \Lambda, \tag{4.19}
$$

where

$$
\begin{aligned}
\Lambda &= \sum_{i=1}^{n-1} \int\limits_{\Delta_i} \big(u_{\tilde z}(\tau) - e_i^\tau\big)\, ds + \sum_{i=1}^{n} \int\limits_{J_i'} \big(T(\tau' - s) e_i^\tau - e_i^\tau\big)\, ds \\
&\quad + \sum_{i=1}^{n} \int\limits_{J_i'} T(\tau' - s)\big(e_i(s) - e_i^\tau\big)\, ds - \sum_{i=1}^{n} \int\limits_{J_i'} \big(T(\tau' - s) u_{\tilde z}(\tau) - u_{\tilde z}(\tau)\big)\, ds \\
&\quad + \sum_{i=1}^{n-1} \int\limits_{\Delta_i} T(\tau' - s)\big(\sigma_{\Delta_i}(s) - u_{\tilde z}(\tau)\big)\, ds + \int\limits_{\tau}^{\tau'} T(\tau' - s)\big(u_{\tilde z}(\tau) - u_{\tilde z}(s)\big)\, ds.
\end{aligned}
$$

Denote by $\Lambda_1, \Lambda_2, \dots, \Lambda_6$ the first, the second, …, the sixth quantity on the right-hand side of the above equality. We have $\|\Lambda_1\| \leqslant 2M \sum_{i=1}^{n-1} |\Delta_i| < \sigma(\tau' - \tau)$ by (4.12), $\|\Lambda_2\| \leqslant \sigma \sum_{i=1}^{n} |J_i'| < \sigma(\tau' - \tau)$ by (4.8), $\|\Lambda_3\| \leqslant \sigma \sum_{i=1}^{n} |J_i'| < \sigma(\tau' - \tau)$ by (4.11), $\|\Lambda_4\| \leqslant \sigma \sum_{i=1}^{n} |J_i'| < \sigma(\tau' - \tau)$ by (4.10), $\|\Lambda_5\| \leqslant 2M \sum_{i=1}^{n-1} |\Delta_i| < \sigma(\tau' - \tau)$ by (4.12), $\|\Lambda_6\| < \sigma(\tau' - \tau)$ by (4.11), and hence

$$
\|\Lambda\| < 6\sigma(\tau' - \tau) < \frac{\varepsilon}{4}(\tau' - \tau), \tag{4.20}
$$

since $\sigma < \varepsilon/24$ by (4.6). Furthermore, as $|J_i| = \lambda_i(\tau' - \tau)$, $i = 1, \dots, n$, where $\lambda_1 + \dots + \lambda_n = 1$, in view of (4.4) we have

$$
\begin{aligned}
\left\| \sum_{i=1}^{n} \int\limits_{J_i} \big(e_i^\tau - u_{\tilde z}(\tau)\big)\, ds \right\| &= \left\| \sum_{i=1}^{n} e_i^\tau |J_i| - \sum_{i=1}^{n} u_{\tilde z}(\tau) |J_i| \right\| \\
&= \left\| \sum_{i=1}^{n} \lambda_i e_i^\tau - u_{\tilde z}(\tau) \right\| (\tau' - \tau) < \frac{\varepsilon}{4}(\tau' - \tau).
\end{aligned} \tag{4.21}
$$

From (4.19), by virtue of (4.20) and (4.21), Claim 2 follows.

The desired smooth mild solution $x : [\tau, \tau'] \to \mathbb{E}$ of $(C_{A,F,\xi_0,\tau})$ satisfying (i)–(v) will now be given by Theorem 3.4. To this end observe that $\xi_0 \in B(z(\tau), R/3)$, from the assumption, and that, by Claim 1 and (4.16), the function $y_\eta : [\tau, \tau'] \to \mathbb{E}$ is a smooth η-solution of $(C_{A,F,\xi_0,\tau})$ satisfying $\|y_\eta(t) - z(t)\| < R/3$ for each $t \in [\tau, \tau']$. Moreover we have $\eta < \frac{R}{9} e^{-K|I|}$, because

$$
\eta = 2KM(\tau' - \tau)^2 < 2KM(\tau' - \tau) < \sigma e^{-K|I|},
$$

by (4.7), and $\sigma < \varepsilon/24 < R/9$ by (4.6) and (4.2). Thus by Theorem 3.4 (with $[\tau, \tau']$ and σ in the place of I and ε, and $\xi = \xi_0$) there exists a smooth mild solution $x : [\tau, \tau'] \to \mathbb{E}$ of $(C_{A,F,\xi_0,\tau})$, with pseudoderivative $u_x \in C([\tau, \tau'], \mathbb{E})$, which satisfies (i), (ii) and furthermore:

$$\|x(t) - y_\eta(t)\| \leqslant q(t) + \sigma(t - \tau), \quad t \in [\tau, \tau'], \tag{4.22}$$

$$\|u_x(t) - \omega(t)\| \leqslant Kq(t) + p(t) + \sigma, \quad t \in [\tau, \tau'], \tag{4.23}$$

where, by virtue of (4.17) and (4.18),

$$p(t) \leqslant 2KM(\tau' - \tau), \qquad q(t) < \sigma(\tau' - \tau), \quad t \in [\tau, \tau']. \tag{4.24}$$

Claim 3. *The smooth mild solution $x : [\tau, \tau'] \to \mathbb{E}$ of $(C_{A,F,\xi_0,\tau})$, with pseudoderivative $u_x \in C([\tau, \tau'], \mathbb{E})$, satisfies* (i)–(v).

It has been shown that x verifies (i) and (ii).

(iii) As $q(t) < \sigma(\tau' - \tau)$ by (4.24), and $\sigma < \varepsilon/4$ by (4.6), then (4.22) furnishes

$$\|x(\tau') - y_\eta(\tau')\| < 2\sigma(\tau' - \tau) < \frac{\varepsilon}{2}(\tau' - \tau).$$

From the latter and Claim 2 it follows

$$\|x(\tau') - \tilde{z}(\tau')\| \leqslant \|x(\tau') - y_\eta(\tau')\| + \|y_\eta(\tau') - \tilde{z}(\tau')\| < \varepsilon(\tau', \tau),$$

proving (iii).

(iv) As $x(\tau) = \xi_0 = \tilde{z}(\tau)$, in view of (4.7) and (4.2) we have

$$\|x(t) - \tilde{z}(t)\| \leqslant \int_\tau^{\tau'} \|T(t-s)\|(\|u_x(s)\| + \|u_{\tilde{z}}(s)\|)\,ds \leqslant 2M(\tau' - \tau) < \sigma, \quad t \in [\tau, \tau'],$$

and thus (iv) holds, since $\sigma < \varepsilon$ by (4.6).

(v) As $x(\tau) = \xi_0 = \tilde{z}(\tau)$, in view of (4.9) for each $t \in [\tau, \tau']$ we have

$$\|x(t) - \tilde{z}(\tau)\| \leqslant \|T(t-\tau)\tilde{z}(\tau) - \tilde{z}(\tau)\| + \int_\tau^t \|T(t-s)\|\|u_x(s)\|\,ds < \sigma + M(\tau' - \tau).$$

Since $\sigma < \delta/2$ by (4.6) and $M(\tau' - \tau) < \delta/2$ by (4.7) and (4.2), it follows that

$$\|x(t) - \tilde{z}(\tau)\| < \delta, \quad t \in [\tau, \tau'].$$

Let $t \in J_i'$, where $1 \leqslant i \leqslant n$. Then,

$$\begin{aligned}\|u_x(t) - e_i^\tau\| &\leqslant \|u_x(t) - \omega(t)\| + \|\omega(t) - e_i^\tau\| \\ &\leqslant Kq(t) + p(t) + \sigma + \|e_i(t) - e_i^\tau\|, \quad \text{by (4.23), (4.13)} \\ &\leqslant K\sigma(\tau' - \tau) + 2KM(\tau' - \tau) + 2\sigma, \quad \text{by (4.24), (4.11).}\end{aligned}$$

Hence

$$\|u_x(t) - e_i^\tau\| < \delta. \tag{4.25}$$

In fact $\sigma < \varepsilon < 1$ and $M > 1$, by (4.6) and (4.2), and thus in view of (4.7) one has $K\sigma(\tau' - \tau) + 2KM(\tau' - \tau) + 2\sigma < 3KM(\tau' - \tau) + 2\sigma < 4\sigma$, which implies (4.25), being $\sigma < \delta/4$ by (4.6). It is evident that $t \in [\tau, \tau + \delta)$, since $t - \tau < \tau' - \tau < \sigma/2$ by (4.7) and (4.2), and $\sigma < \delta$ by (4.6).

Summarizing, for $t \in J_i'$, $i = 1, \dots, n$, we have

$$t \in [\tau, \tau + \delta), \quad \|x(t) - \tilde{z}(\tau)\| < \delta, \qquad \|u_x(t) - e_i^\tau\| < \delta.$$

Since, in addition, $u_x(t) \in F(t, x(t))$, $t \in [\tau, \tau']$, by virtue of (4.5) it follows that

$$0 \leqslant d_F(t, x(t), u_x(t)) < \alpha/2, \quad t \in J_i',\ i = 1, \dots, n. \tag{4.26}$$

On the other hand F is bounded by M and hence, by Lemma 2.4, $0 \leqslant d_F(t, x(t), u_x(t)) \leqslant M^2$ for every $t \in [\tau, \tau']$. In view of this and (4.26) one has

$$\begin{aligned}\int_\tau^{\tau'} d_F(t, x(t), u_x(t))\,dt &= \sum_{i=1}^n \int_{J_i'} d_F(t, x(t), u_x(t))\,dt + \sum_{i=1}^{n-1} \int_{\Delta_i} d_F(t, x(t), u_x(t))\,dt \\ &< \frac{\alpha}{2} \sum_{i=1}^n |J_i'| + M^2 \sum_{i=1}^{n-1} |\Delta_i| < \frac{\alpha}{2}(\tau' - \tau) + \frac{\sigma}{2}(\tau' - \tau), \quad \text{by (4.12).}\end{aligned}$$

As $\sigma < \alpha$ by (4.6), it follows

$$\int_{\tau}^{\tau'} d_F\big(t, x(t), u_x(t)\big)\,dt < \alpha\big(\tau' - \tau\big).$$

Hence (v) holds. This completes the proof. $\square$

Theorem 4.1. *Let A and $F : I \times \mathbb{E} \to \mathcal{C}(\mathbb{E})$ satisfy (h_1)–(h_4). Let $z : I \to \mathbb{E}$ be a smooth mild solution of $(C_{A,F,a})$, with pseudoderivative $u_z : I \to \mathbb{E}$, and let $R = R_z > 0$, $K = K_z > 0$ correspond to z. Let $\varepsilon > 0$ and $\alpha > 0$ be given. Then there exists a mild solution $x : I \to \mathbb{E}$ of $(C_{A,F,a})$, with pseudoderivative $u_x \in L^\infty(I, \mathbb{E})$, such that*

$$\|x - z\|_I \leqslant \varepsilon, \tag{4.27}$$

$$\int_I d_F\big(t, x(t), u_x(t)\big)\,dt \leqslant \alpha |I|. \tag{4.28}$$

Proof. Without loss of generality we assume

$$K > 1, \qquad M > 1, \qquad R < 1, \qquad 0 < \varepsilon < 1. \tag{4.29}$$

Denote by Λ the set of all mild solutions $x : [t_0, \tau_x] \to \mathbb{E}$ of $(C_{A,F,a})$ (where $\tau_x \in (t_0, t_1]$ depends on x), with corresponding pseudoderivative $u_x \in L^\infty(I, \mathbb{E})$, satisfying the following conditions:

(j) $$x(t) = T(t - t_0)a + \int_{t_0}^{t} T(t - s)u_x(s)\,ds, \quad t \in [t_0, \tau_x],$$

$$u_x(t) \in F\big(t, x(t)\big), \quad t \in [t_0, \tau_x] \text{ a.e.},$$

(jj) $$\big\|x(\tau_x) - z(\tau_x)\big\| \leqslant \varepsilon \frac{R}{18} e^{-(2+K)(t_1 - \tau_x)},$$

(jjj) $$\big\|x(t) - z(t)\big\| \leqslant \varepsilon, \quad t \in [t_0, \tau_x],$$

(jv) $$\int_{t_0}^{\tau_x} d_F\big(t, x(t), u_x(t)\big)\,dt \leqslant \alpha(\tau_x - t_0).$$

Set

$$\varepsilon' = \varepsilon \frac{R}{18} e^{-(2+K)(t_1 - t_0)}. \tag{4.30}$$

An application of Lemma 4.1 (with $\tau = t_0$, $\xi_0 = z(t_0) = a$, $z = \tilde{z}$ and ε replaced by ε') furnishes a smooth mild solution $x : [t_0, \tau_x] \to \mathbb{E}$ of $(C_{A,F,a})$, for some $\tau_x \in (t_0, t_1]$, with $\tau_x - t_0 < 1$, satisfying properties (i)–(v) of Lemma 4.1. Clearly x verifies (j) and (jv). Furthermore, (iii) implies (jj), since $\varepsilon'(\tau_x - t_0) < \varepsilon' < (\varepsilon R/18)e^{-(2+K)(t_1 - \tau_x)}$, while (jjj) follows from (iv) since $\varepsilon' < \varepsilon$, being $R < 1$ by (4.29). Hence Λ is nonempty.

For $y_\alpha, y_\beta \in \Lambda$, where $y_\alpha : [t_0, \tau_\alpha] \to \mathbb{E}$ and $y_\beta : [t_0, \tau_\beta] \to \mathbb{E}$, define $y_\alpha \prec y_\beta$ to mean $\tau_\alpha \leqslant \tau_\beta$ and $y_\alpha(t) = y_\beta(t)$, for every $t \in [t_0, \tau_\alpha]$. Λ equipped with the relation $\prec$ is a partially ordered set.

Claim 1. *Each totally ordered subset $\{y_\gamma\}_{\gamma \in \Gamma}$ of Λ has an upper bound.*

Let

$$\tau^* = \sup\{\tau_\gamma \mid \gamma \in \Gamma\}.$$

If $\tau^* = \tau_\alpha$ for some $\alpha \in \Gamma$ then y_α is an upper bound of $\{y_\gamma\}_{\gamma \in \Gamma}$. Suppose that $\tau_\gamma < \tau^*$ for every $\gamma \in \Gamma$ and consider an increasing sequence $\{\tau_{\gamma_n}\}$, $\gamma_n \in \Lambda$, converging to τ^*. We write τ_n, y_n, u_n to denote τ_{γ_n}, y_{γ_n}, $u_{y_{\gamma_n}}$. Define $y^* : [t_0, \tau^*) \to \mathbb{E}$ and $u^* : [t_0, \tau^*) \to \mathbb{E}$ respectively by

$$y^*(t) = y_n(t), \quad t \in [t_0, \tau_n] \quad \text{and} \quad u^*(t) = u_n(t), \quad t \in [t_0, \tau_n] \text{ a.e.}$$

Clearly y^* is well defined and as in the proof of Claim 1 of Theorem 3.2 one can show that y^*, with corresponding pseudoderivative u^*, is a mild solution of $(C_{A,F,a})$ on $[t_0, \tau^*)$. Moreover $\{y_n(\tau_n)\}$ is a Cauchy sequence and so y^* admits a continuous extension say y^*, defined on the closed interval $[t_0, \tau^*]$, which is a mild solution of $(C_{A,F,a})$. Since y^* agrees with y_n on $[t_0, \tau_n]$, it follows that y^* satisfies (j)–(jv) on every interval $[t_0, \tau_n]$, and so on $[t_0, \tau^*]$. Therefore $y^* \in \Lambda$. It is evident that $y_n \prec y^*$ for any $n \in \mathbb{N}$. Since $\tau_n \to \tau^*$ as $n \to \infty$, for each $y_\gamma \in \Gamma$ there exists $n \in \mathbb{N}$ such that $y_\gamma \prec y_n$. Hence $y_\gamma \prec y^*$ for every $y_\gamma \in \Gamma$, i.e. $y^* \in \Lambda$ is an upper bound of Γ and Claim 1 holds.

By Zorn's lemma Λ contains a maximal element, say $x : [t_0, \tau] \to \mathbb{E}$, with corresponding $u_x : [t_0, \tau] \to \mathbb{E}$, for some $\tau \in (t_0, t_1]$.

Claim 2. $\tau = t_1$.

Suppose, by contradiction, that $\tau < t_1$. As $x \in \Lambda$, by (jj) we have

$$\|x(\tau) - z(\tau)\| \leqslant \varepsilon \frac{R}{18} e^{-(2+K)(t_1-\tau)}. \tag{4.31}$$

Moreover (4.30) and (4.31) imply, respectively,

$$\varepsilon' < \frac{R}{18}, \qquad \|x(\tau) - z(\tau)\| < \frac{R}{18} e^{-(2+K)(t_1-\tau)}, \tag{4.32}$$

since $\varepsilon < 1$ by (4.29). Fix $\tau_1 \in (\tau, t_1)$, with $\tau_1 - \tau < \frac{\varepsilon}{8M}$, sufficiently close to τ so that we have

$$\|T(t-\tau)x(\tau) - x(\tau)\| < \frac{\varepsilon}{4}, \qquad \|T(t-\tau)z(\tau) - z(\tau)\| < \frac{\varepsilon}{4}, \quad t \in [\tau, \tau_1]. \tag{4.33}$$

In view of (4.32), the assumptions of Theorem 3.4 are satisfied (with ε, t_0, I replaced by ε', τ, $[\tau, \tau_1]$, and $y_\eta(t) = z(t)$, $t \in [\tau, \tau_1]$, $\xi_0 = z(\tau)$, $\xi = x(\tau)$), and so there exists a smooth mild solution $\tilde{z} : [\tau, \tau_1] \to \mathbb{E}$ of $(C_{A,F,x(\tau),\tau})$, with pseudoderivative $u_{\tilde{z}} \in C([\tau, \tau_1], \mathbb{E})$, such that

$$\|\tilde{z}(t) - z(t)\| \leqslant \|x(\tau) - z(\tau)\| e^{K(t-\tau)} + \varepsilon'(t - \tau), \quad t \in [\tau, \tau_1], \tag{4.34}$$

$$\|u_{\tilde{z}}(t) - u_z(t)\| \leqslant K \|x(\tau) - z(\tau)\| e^{K(t-\tau)} + \varepsilon', \quad t \in [\tau, \tau_1]. \tag{4.35}$$

Clearly $\|x(\tau) - z(\tau)\| < R/18$ by (4.32), and thus (4.34) implies

$$\|\tilde{z}(t) - z(t)\| < \frac{R}{9}, \quad t \in [\tau, \tau_2], \tag{4.36}$$

for some $\tau_2 \in (\tau, \tau_1)$ sufficiently close to τ. Then by virtue of Lemma 4.1 (with ε, ξ_0, r replaced by ε', $x(\tau)$, $R/9$), taking into account (4.36), there exists a smooth mild solution $\tilde{x} : [\tau, \tau'] \to \mathbb{E}$ of $(C_{A,F,x(\tau),\tau})$ with pseudoderivative $u_{\tilde{x}} \in C([\tau, \tau'], \mathbb{E})$, for some $\tau' \in (\tau, \tau_2)$, such that $\tilde{x}$ has graph contained in the tube $N(z, \frac{2}{3}R)$ and, moreover, $\tilde{x}$ satisfies the following properties:

$$\begin{aligned}
&\tilde{x}(t) = T(t-\tau)x(\tau) + \int_\tau^t T(t-s)u_{\tilde{x}}(s)\,ds, \quad t \in [\tau, \tau'], \\
&u_{\tilde{x}}(t) \in F(t, \tilde{x}(t)), \quad t \in [\tau, \tau'], \\
&\|\tilde{x}(\tau') - \tilde{z}(\tau')\| < \varepsilon'(\tau' - \tau), \quad t \in [\tau, \tau'], \\
&\|\tilde{x}(t) - \tilde{z}(t)\| < \varepsilon', \quad t \in [\tau, \tau'],
\end{aligned} \tag{4.37}$$

$$\int_\tau^{\tau'} d_F(t, \tilde{x}(t), u_{\tilde{x}}(t))\,dt < \alpha(\tau' - \tau). \tag{4.38}$$

Clearly $\tilde{x}(\tau) = x(\tau) = \tilde{z}(\tau)$. In view of (4.37) and (4.34), we have

$$\|\tilde{x}(\tau') - z(\tau')\| \leqslant \|\tilde{x}(\tau') - \tilde{z}(\tau')\| + \|\tilde{z}(\tau') - z(\tau')\| \leqslant \|x(\tau) - z(\tau)\| e^{K(\tau'-\tau)} + 2\varepsilon'(\tau' - \tau),$$

and thus, by (4.30) and (4.31),

$$\begin{aligned}
\|\tilde{x}(\tau') - z(\tau')\| &\leqslant \frac{\varepsilon R}{18}\left[e^{-(2+K)(t_1-\tau)} e^{K(\tau'-\tau)} + 2(\tau' - \tau) e^{-(2+K)(t_1-t_0)}\right] \\
&< \frac{\varepsilon R}{18} e^{-(2+K)(t_1-\tau')}.
\end{aligned} \tag{4.39}$$

Here the last inequality is valid since the real function

$$f(x) = e^{-(2+K)(t_1-x)} - e^{-(2+K)(t_1-\tau)}e^{K(x-\tau)} - 2(x-\tau)e^{-(2+K)(t_1-t_0)}, \quad x \in [\tau, \tau'],$$

is strictly increasing and vanishes for $x = \tau$.

Define now $\hat{x} : [t_0, \tau'] \to \mathbb{E}$ and $u_{\hat{x}} : [t_0, \tau'] \to \mathbb{E}$ as follows

$$\hat{x}(t) = \begin{cases} x(t), & t \in [t_0, \tau], \\ \tilde{x}(t), & t \in [\tau, \tau'], \end{cases} \qquad u_{\hat{x}}(t) = \begin{cases} u_x(t), & t \in [t_0, \tau], \\ u_{\tilde{x}}(t), & t \in [\tau, \tau']. \end{cases} \tag{4.40}$$

We will show that $\hat{x}$ satisfies (j)–(jv) and so $\hat{x} \in \Lambda$.

(j) By construction $\hat{x} : [t_0, \tau'] \to \mathbb{E}$ is continuous and is a mild solution of $(C_{A,F,a})$, with pseudoderivative $u_{\hat{x}}$.

(jj) Since $\hat{x}(\tau') = \tilde{x}(\tau')$, (4.39) implies

$$\|\hat{x}(\tau') - z(\tau')\| \leqslant \frac{\varepsilon R}{18} e^{-(2+K)(t_1-\tau')}.$$

(jjj) As $\hat{x}(t) = x(t)$ for $t \in [t_0, \tau]$ and $x \in \Lambda$ one has

$$\|\hat{x}(t) - z(t)\| \leqslant \varepsilon, \quad t \in [t_0, \tau]. \tag{4.41}$$

The above inequality remains valid also for $t \in [\tau, \tau']$. In fact, in view of the definition of $\hat{x}$ and $\tilde{x}$, for $t \in [\tau, \tau']$ we have

$$\begin{aligned}
\|\hat{x}(t) - z(t)\| &= \|\tilde{x}(t) - z(t)\| \leqslant \|\tilde{x}(t) - \tilde{x}(\tau)\| + \|\tilde{x}(\tau) - z(\tau)\| + \|z(\tau) - z(t)\| \\
&= \left\| T(t-\tau)x(\tau) + \int_\tau^t T(t-s)u_{\tilde{x}}(s)\,ds - x(\tau) \right\| + \|\tilde{x}(\tau) - z(\tau)\| \\
&\quad + \left\| T(t-\tau)z(\tau) + \int_\tau^t T(t-s)u_z(s)\,ds - z(\tau) \right\| \\
&\leqslant \int_\tau^t \|T(t-s)\| \left(\|u_{\tilde{x}}(s)\| + \|u_z(s)\| \right) ds + \|T(t-\tau)x(\tau) - x(\tau)\| \\
&\quad + \|T(t-\tau)z(\tau) - z(\tau)\| + \|x(\tau) - z(\tau)\|.
\end{aligned}$$

The second and third term of the latter inequality are each one less than $\varepsilon/4$ by (4.33), while the last term is less than $\frac{\varepsilon R}{18} e^{-(2+K)(t_1-\tau)}$ by (jj), as $x : [t_0, \tau] \to \mathbb{E}$ is in Λ. Hence,

$$\|\hat{x}(t) - z(t)\| < 2M(\tau' - \tau) + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon R}{18} e^{-(2+K)(t_1-\tau)} < \varepsilon, \quad t \in [\tau, \tau'], \tag{4.42}$$

as $\tau' - \tau < \tau_1 - \tau < \frac{\varepsilon}{8M}$ and $R < 1$. From (4.41) and (4.42) it follows

$$\|\hat{x}(t) - z(t)\| \leqslant \varepsilon, \quad t \in [t_0, \tau'],$$

and thus $\hat{x}$ satisfies (jjj).

(jv) In view of (4.40), since $x \in \Lambda$ and $\tilde{x}$ verifies (4.38), we have

$$\begin{aligned}
\int_{t_0}^{\tau'} d_F(t, \hat{x}(t), u_{\hat{x}}(t))\,dt &= \int_{t_0}^{\tau} d_F(t, x(t), u_x(t))\,dt + \int_{\tau}^{\tau'} d_F(t, \tilde{x}(t), u_{\tilde{x}}(t))\,dt \\
&< \alpha(\tau - t_0) + \alpha(\tau' - \tau) = \alpha(\tau' - t_0),
\end{aligned}$$

and hence $\hat{x}$ satisfies (jv).

In conclusion $\hat{x} : [t_0, \tau'] \to \mathbb{E}$ is a mild solution of $(C_{A,F,a})$ satisfying (j)–(jv), and thus $\hat{x} \in \Lambda$. As $x \prec \hat{x}$ and $x \neq \hat{x}$ a contradiction follows. Consequently $\tau = t_1$ and Claim 2 is proved.

As x lies in Λ and, by Claim 2, x is defined all over I, then the map $x : I \to \mathbb{E}$ is the required mild solution of the Cauchy problem $(C_{A,F,a})$ satisfying (4.27) and (4.28). This completes the proof. $\square$

5. The bang-bang property

With the help of the technical results established in the previous section we are now ready to prove the following bang-bang property.

Theorem 5.1. *Let A and $F : I \times \mathbb{E} \to \mathcal{C}(\mathbb{E})$ satisfy (h_1)–(h_4) and let $a \in \mathbb{E}$. Then the set $\mathcal{M}_{A,\operatorname{ext} F,a}$ is residual in $\mathcal{M}_{A,F,a}$. In particular $\overline{\mathcal{M}}_{A,\operatorname{ext} F,a} = \mathcal{M}_{A,F,a}$, where the closure is in the metric of uniform convergence.*

Proof. For $k \in \mathbb{N}$ set

$$\mathcal{M}_k = \left\{ x \in \mathcal{M}_{A,F,a} \;\middle|\; \int_I d_F\big(t, x(t), u_x(t)\big)\,dt < 2^{-k}|I| \right\}.$$

Claim 1. *$\mathcal{M}_k$ is open in $\mathcal{M}_{A,F,a}$.*

It suffices to show that if $\{x_n\} \subset \mathcal{M}_{A,F,a} \setminus \mathcal{M}_k$ is a sequence which converges uniformly to $x \in \mathcal{M}_{A,F,a}$ then $x \in \mathcal{M}_{A,F,a} \setminus \mathcal{M}_k$. Consider the sequence $\{u_{x_n}\}$, where u_{x_n} corresponds to x_n. Clearly $\|u_{x_n}(t)\| \leqslant M$, $t \in I$, and thus $\{u_{x_n}\}$ is contained in a closed ball of $L^2(I,\mathbb{E})$. As $L^2(I,\mathbb{E})$ is reflexive there exists a subsequence, say $\{u_{x_n}\}$, which converges weakly in $L^2(I,\mathbb{E})$ and so in $L^1(I,\mathbb{E})$ to some $v \in L^2(I,\mathbb{E})$. By Mazur's lemma there exists a sequence of convex combinations, say

$$\left\{ \sum_{i=n_k}^{n_{k+1}-1} \lambda_i^k u_{x_i} \right\}, \quad \text{where} \sum_{i=n_k}^{n_{k+1}-1} \lambda_i^k = 1,\ \lambda_i^k \geqslant 0,\ n_1 < n_2 < \cdots,$$

which converges to v in $L^1(I,\mathbb{E})$. Clearly for each $i \in \mathbb{N}$,

$$x_i(t) = T(t-t_0)a + \int_{t_0}^{t} T(t-s)u_{x_i}(s)\,ds, \quad t \in I,$$

and thus

$$\sum_{i=n_k}^{n_{k+1}-1} \lambda_i^k x_i(t) = T(t-t_0)a + \int_{t_0}^{t} T(t-s)\left(\sum_{i=n_k}^{n_{k+1}-1} \lambda_i^k u_{x_i}(s) \right) ds, \quad t \in I,$$

from which, letting $k \to \infty$, one has

$$x(t) = T(t-t_0)a + \int_{t_0}^{t} T(t-s)v(s), \quad t \in I.$$

On the other hand,

$$x(t) = T(t-t_0)a + \int_{t_0}^{t} T(t-s)u_x(s), \quad t \in I,$$

and hence, by Remark 2.2, it follows that $u_x = v$. As $x_n \to x$ uniformly on I, and $u_{x_n} \to u_x$ weakly in $L^1(I,\mathbb{E})$, by Lemma 2.3 we have

$$\int_I d_F\big(t, x(t), u_x(t)\big)\,dt \geqslant \limsup_{n\to\infty} \int_I d_F\big(t, x_n(t), u_{x_n}(t)\big)\,dt \geqslant 2^{-k}|I|.$$

Hence $x \in \mathcal{M}_{A,F,a} \setminus \mathcal{M}_k$, proving Claim 1.

Claim 2. *$\mathcal{M}_k$ is dense in $\mathcal{M}_{A,F,a}$.*

Let $y \in \mathcal{M}_{A,F,a}$ and let $\varepsilon > 0$. By Theorem 3.5, there exists a smooth mild solution $z \in \mathcal{M}_{A,F,a}$ such that $\|z-y\|_I < \varepsilon/2$. By Theorem 4.1, there exists a mild solution $x \in \mathcal{M}_{A,F,a}$, with $\|x-z\|_I < \varepsilon/2$, satisfying

$$\int_I d_F\big(t, x(t), u_x(t)\big)\,dt < 2^{-k}|I|.$$

From the latter, as $\|x - y\|_I < \varepsilon$, it follows that $\mathcal{M}_k$ is dense in $\mathcal{M}_{A,F,a}$, proving Claim 2.

Set

$$\mathcal{M}_0 = \bigcap_{k=1}^{\infty} \mathcal{M}_k,$$

and observe that $\mathcal{M}_0$ is residual in $\mathcal{M}_{A,F,a}$, a nonempty complete metric space. By Baire's theorem, the set $\mathcal{M}_0$ is dense in $\mathcal{M}_{A,F,a}$. Moreover, any $x \in \mathcal{M}_0$ satisfies $\int_I d_F(t, x(t), u_x(t))\,dt < 2^{-k}|I|$, for all $k \in \mathbb{N}$, and so $d_F(t, x(t), u_x(t)) = 0$, $t \in I$ a.e. By Lemma 2.3 it follows that $u_x(t) \in \operatorname{ext} F(t, x(t))$, $t \in I$ a.e. Therefore $x \in \mathcal{M}_{A,\operatorname{ext} F,a}$ and thus

$$\mathcal{M}_0 \subset \mathcal{M}_{A,\operatorname{ext} F,a} \subset \mathcal{M}_{A,F,a}.$$

Since $\mathcal{M}_0$ is dense in $\mathcal{M}_{A,F,a}$, we have $\overline{\mathcal{M}}_{A,\operatorname{ext} F,a} = \mathcal{M}_{A,F,a}$. This completes the proof. □

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